Appendix 1

1 Definitions

The players are three legislators, $N = \{m, M_1, M_2\}$. By majority voting, they choose policy outcomes from the convex policy space $X \subseteq \Re^2$. Legislators have weak preferences denoted $R_i,\ i \in N$, that are representable as Euclidean distance utility functions,

$$u_i(x) = -d(x, x_i) = -\sqrt{(x_1 - x_{i1})^2 + (x_2 - x_{i2})^2},$$

where $x_i \in X$ is legislator i's ideal point, and the numeric subscripts index the coordinates. Therefore, the terms x and $(x_1, x_2) \in X$ are interchangeable. We use the standard notation $R_i(x) = \{y \in X : y\ R_i\ x\}$ and $R_i^{-1}(x) = \{y \in X : x\, R_i\, y\}$ and the analogously defined sets P_i, I_i and P_i^{-1}, for $i \in N$.

Definition 1 Pareto set: *The Pareto set, PS, is defined as* $\{x \in X : \tilde{\ }\exists y \in X$ *s.t.* $yR_ix\ \forall i \in N\ \&\ yP_ix\ \exists i \in N\}$

Definition 2 Strategies: *A strategy profile to the extensive form game is an ordered tuple* $\left\langle b(q), a(b;q), \{v_i^1(x,y;q), v_i^2(x;q), v_i^3(x;q)\}_{i\in N}\right\rangle$ *where* $b \in X$ *is* M_1*'s proposal,* $a : X \to X$ *is* m*'s amendment function,* $v_i^1 : X \times X \to \{0,1\}$ *and* $v_i^2 : X \to \{0,1\}$ *and* $v_i^3 : X \to \{0,1\}$ *are the voting strategies in the three voting stages of the agenda for* $i \in N$.[1][2]

Definition 3 Equilibrium: *Let* $R(x,y) = |i \in N : x\, R_i\, y|$. *Then a subgame perfect Nash equilibrium in weakly undominated strategies*[3] *to the game is a tuple:* $\left\langle b^*(q), a^*(b;q), \{v_i^{*1}(x,y;q), v_i^{*2}(x;q), v_i^{*3}(x;q)\}_{i\in N}\right\rangle$ *s.t.* $\forall i \in N$,

$$v_i^{3*}(x;q) = \begin{cases} 1 \text{ if } xR_iq \\ 0 \text{ otherwise} \end{cases}$$

$$v_i^{2*}(x;q) = \begin{cases} 1 \text{ if } xR_iq \\ 0 \text{ otherwise} \end{cases}$$

---

[1] The domain of the first period voting functions is the Cartesian product of two policy spaces, because the function must be defined for all possible proposals and amendments. The remaining voting functions have as domain only the policy space, because the status quo is automatically chosen as the alternative to the winning policy in the second and third stage votes.

[2] The term $v(x,y) = 1$ is interpreted to mean player i votes for x over y and $v_i(x) = 1$ is interpreted to mean player i votes for x over q.

[3] This concept is stronger than Nash equilibrium in two ways. Subgame perfection requires that players anticipate future behavior that constitutes Nash equilibrium behavior of the future subgame. The requirement that strategies be weakly undominated prevents players from unanimously voting for an undesirable outcome, just because any single deviation, will not effect the outcome.

$$v_i^{1*}(x,y;q) = \begin{cases} 1 \text{ if } xP_iy \text{ and } R(x,q) \geq 2 \text{ and } R(y,q) \geq 2 \\ 1 \text{ if } xP_iq \text{ and } R(x,q) \geq 2 \text{ and } R(y,q) < 2 \\ 1 \text{ if } qP_iy \text{ and } R(x,q) < 2 \text{ and } R(y,q) \geq 2 \\ 0 \text{ otherwise} \end{cases}$$

$$a^*(b;q) \in \arg\max\{u_m(x(b,a;q))\}$$

$$b^*(q) \in \arg\max\{u_{M_1}(x(b,a^*(b;q);q))\}$$

$$\text{where } x(a,b;q) = \begin{cases} a \text{ if } a \in W(q) \cap W(b) \text{ and } b \in W(q) \\ a \text{ if } a \in W(q) \text{and } b \notin W(q) \\ b \text{ if } b \in W(q) \text{ and } a \notin W(q) \\ q \text{ if } b \notin W(q) \text{ and } a \notin W(q) \end{cases}.$$

The complexity of $v_i^1(x,y)$ is due to the need to capture sophisticated voting. The various conditions in the definition characterize the possible subsequent behavior that determines the sophisticated equivalents at the first of two stages of voting. The variable x captures the equilibrium play of the voting subgames for each possible (a,b) pair. The arguments of x, a, and b are suppressed when the meaning is clear.

Definition 4 *$q-power$: A player i is said to be q-powerful iff $x^*(q)\, R_i\, q$.*

Definition 5 *$u-power$: A player i is said to be u-powerful iff $x^*(q)\, R_i\, q\ \forall q \in PS$.*[4]

The notion of q-power makes clear a form of power which is status quo dependent. In contrast the stronger notion of u-power refers to a notion of power that holds for all possible status quos.

Definition 6 Maximal elements: *Let $C_i(A)$ denote the set of i's most preferred alternatives from the set $A \subset X$.*[5]

---

[4] Definitions of power can be straightforwardly applied to the set of majority party members M by requiring that the relevant condition apply for for all $i \in M$.

[5] See for example Figure 3b. Let $i = m$ and let the set A be $W(q) \cap W(b)$, or the two cross-hatched bullet heads. Then $C_i(A) = \{a^*, \widehat{a}\}$.

2 Lemmas

Proofs of the propositions are simplified by establishing six lemmas. The lemmas require only that R_i be complete, reflexive, transitive, strictly convex and continuous. That is, circular indifference curves are not essential for the formal argument. However, to insure that the Pareto set is a two dimensional surface and simplify the analysis, we assume in the propositions that preferences are Euclidean as stated above.

Lemma 1 establishes that the equilibrium proposals (a^*, b^*) and outcome (x^*), if they exist, can be assumed to meet three win-set conditions as illustrated, for example, in Figure 3b. Lemma 2 shows that any proposal outside the Pareto set is dominated by a proposal within the Pareto set. Lemmas 3 and 4 show existence and uniqueness of an equilibrium, respectively. Lemma 5 states that equilibrium outcome x^* is continuous in the status quo q. Lemma 6 establishes a result of greater technical than substantive importance: if q-power exists for a point, then it exists for a neighborhood around that point.

Lemma 1 states that we may assume optimal proposals satisfy reasonable conditions. The behavior that is implicit but central is that of optimizing agenda construction (Austen-Smith 1987) under expectations of sophisticated voting (Farquaharson 1969; McKelvey and Niemi 1976). Note that, if a proposal is to be outcome-consequential in an finite, binary agenda, it must be preferred by a majority to all subsequent proposals it may encounter in the course of voting.

Lemma 1 Conditions for best-response proposals: *(a) $\forall q \in PS$,if a best response $a(b)$ exists, then at least one exists satisfying the condition $a(b) \in W(q)$ $\forall b$. (b) $\forall q \in PS$ and $\forall b \in W(q)$, if a best response $a(b)$ exists, then at least one exists satisfying the condition $a(b) \in W(b)$. (c) Assume a best response $a(b)$ exists, then $\forall q \in PS$,if a best response b exists, then at least one exists satisfying the condition $b \in W(q)$.*

> **Proof**: (a) and (b): Suppose $b \in W(q)$. Since $R_m(b) \cap W(q) \cap W(b) \neq \emptyset$, for all $a \notin W(q)$, $\exists a' \in R_m(b) \cap W(q) \cap W(b)$. Since $x(a,b,q) = b$ and $x(a',b,q) = a'$ and $a' \in R_m(b)$ this implies $x(a',b,q) R_m x(a,b,q)$. Suppose $b \notin W(q)$ then $x \in \{a,q\}$. Since $R_m(q) \cap W(q) \neq \emptyset$, for all $a \notin W(q)$, $\exists a' \in R_m(b) \cap W(q)$. Since $x(a,b,q) = q$ and $x(a',b,q) = a'$ and $a' \in R_m(q)$ this implies $x(a',b,q) R_m x(a,b,q)$. This establishes (a) and (b).
>
> (c) Assume a best response $a(b)$ exists and assume that a best response b exists. First observe that if $b \in W(q)$ results (a) and (b) imply that m has a best response in the set: $W(q) \cap W(b)$ If $b \notin W(q)$ then by (a) m has a best response in the set $W(q)$. It remains to establish that $\exists b' \in W(q)$ s.t. $C_m(W(q) \cap W(b')) \in R_{M_1}(C_m(W(q)))$. Since a best response $a(b') \in W(b')$ exists, we know that $a(b') \in$

$R_j(b')\ \exists j \in M$. Moreover, we know that $C_m(W(q) \cap W(b')) \subset C_m(W(q))$. This implies that if $b' \in I_{M_1}(C_m(W(q)))$, M_1 will do no worse than under $b \notin W(q)$. To see that $I_{M_1}(C_m(W(q)) \cap W(q)) \neq \emptyset$, observe that $C_m(W(q)) \in I_{M_1}(C_m(W(q)) \cap W(q))$, and $C_m(W(q)) \cap W(q)$ was assumed to exist.■

Lemma 2 allows generalization of the Aldrich-Rohde game to $\Re^n$ as long as the Euclidean preference assumption is maintained, as the result implies that all proposals will lie on the Pareto Set -a two dimensional surface for three players with Euclidean preferences, regardless of how high the dimensionality of the policy space is. The lemma also serves to justify considering only policies lying in the Pareto Set. The result is otherwise obvious and unimportant. It says that, for $X \subset \Re^n$, any proposals a and b not in the Pareto set are dominated by some proposal in the Pareto set.

Lemma 2 Dominated alternatives: *If $X \subset \Re^n$ is convex, and R_i then: (1) For any mapping $a : X \times PS \to X$ in which $a(b;q) \in \{W(q) \cap W(b)\} - PS$ for some pair $(q,b) \in PS \times X$, an alternative mapping $a' : X \times PS \to PS$ exists in which $a'(b;q) \in W(q)$ and $a'(b,q)\, R_m\, a(b,q)$ on $X \times PS$ and $a(b,q) \notin PS$ implies $a'(b,q)\, P_m\, a(b,q)$. (2) For any mapping $b : PS \to X$ in which $b(q) \in W(q) - PS$ for some $q \in PS$, an alternative mapping $b' : PS \to PS$ exists in which $b'(q) \in W(q)$ and $a(b';q)\, R_{M_1}\, a(b;q)$ and $b(q) \notin PS$ implies $a(b';q)\, P_{M_1}\, a(b;q)$.*

Proof. Available by request.

Lemmas 1 and 2 insure that the search for equilibrium proposals must be confined to a compact set. This fact and the continuity of preferences allow us to establish existence for an arbitrary parameterization of complete, reflexive, strictly convex, and continuous preferences and status quo. Lemmas 3 and 4 follow from Austen-Smith (1987), but the proof of Lemma 3 uses a different technique that transports more easily to subsequent results and illustrates the intuition behind strategic action in the game. The central intuition to the proof is that since preferences are continuous the voting behavior in the voting stages induces m's optimal amendment to vary continuously with the proposal. Difficulty arrises because the set of optimal amendments is not necessarily single-valued. Nonetheless, we can show that the proposal problem M_1 faces is sufficiently well behaved to insure an optimal proposal exists.

Lemma 3 Existence: *For all $q \in PS$, a subgame perfect Nash equilibrium in weakly undominated strategies exists.*

Proof. Reasoning by backward induction in three steps the result is demonstrated by showing that all subgames—voting, m's amendment, and M_1's bill—have an equilibrium.

Step 1: voting stages. For each of the two possible final-vote stages and $\forall b, a, q \in PS$, each player has a weakly dominant strategy because preferences are complete and the choice is binary. Therefore, voting decisions at the last stage are well defined, and sophisticated equivalents (Mckelvey and Niemi 1976) exist for the first stage of voting. This fact, the completeness of preferences, and the fact that the decision at the first stage of voting is binary, imply that each agent also has a weakly dominant strategy in the first stage of voting. The strategy is to vote as specified in the definition of equilibrium. It follows also that, as in Def. 3, x is the outcome function on which proposal behavior is conditioned.

Step 2: We show that given $q \in PS$ and $b \in PS \cap W(q)$, m's best response is well defined i.e. $a^*(b,q)$ exists. Assume $q \in PS$ and $b \in PS \cap W(q)$.

Let $C_{ijk}(b,q) := R_i(b) \cap R_j(q) \cap R_k(q) \cap PS$ for $i,j,k \in N$. Since preferences are strictly convex $C_{ijk}(b,q)$ is convex valued. Since preferences are continuous and PS is compact, $C_{ijk}(b,q)$ is compact valued. Let $E_{ijk} := \{b,q \in PS \times PS : C_{ijk}(b,q) \neq \emptyset\}$ and let $E_{ijk}(q) := \{b \in PS : C_{ijk}(b,q) \neq \emptyset\}$. Let $a_{ijk}(b,q) := \arg\max_{a \in C_{ijk}(b,q)} u_m(x(a,b,q))$. As long as $(b,q) \in E_{ijk}$, $a_{ijk}(b,q)$ exists and is unique. This is true because $(b,q) \in E_{ijk}$ implies that $C_{ijk}(b,q)$ is non empty compact and convex, and $a \in C_{ijk}(b,q)$ implies that $x(a,b,q) = a$ so that $u_m(x(a,b,q))$ is continuous and strictly quasi concave in a. Moreover by the Theorem of the Maximum $a_{ijk}(b,q)$ is a continuous function of b,q on E_{ijk}, and thus $a_{ijk}(b;q)$ is a continuous function of b on $E_{ijk}(q)$. Let $A(b,q) := \{a_{112}(b,q), a_{11m}(b,q), a_{12m}(b,q), a_{212}(b,q), a_{21m}(b,q), a_{22m}(b,q)\}$ (where $i,j,k = t$ denotes M_t). The fact that $b \in W(q)$ implies $W(b) \cap W(q) \neq \emptyset$. Given this, $b \in W(q)$ implies $C_{112}(b,q) \cup C_{11m}(b,q) \cup C_{12m}(b,q) \cup C_{212}(b,q) \cup C_{21m}(b,q) \cup C_{22m}(b,q) \neq \emptyset$. This implies that $A(b,q) \neq \emptyset$ as at least one of the relevant $a_{ijk}(b,q)$'s will exist. Moreover, by lemma 1, m will choose a in $W(q) \cap W(b)$ and $x(a,b,q) = a$, thus m will choose from $A(b,q)$. So m's best response correspondence is $A^*(b,q) =: \arg\max_{a \in A(b,q)} u_m(a)$. Note that $A^*(b,q) \neq \emptyset$ follows from the fact that there are a finite number of points $a_{ijk}(b,q)$ in $A(b,q)$. Thus m's best response $A^*(b,q)$ is well defined. We denote the value function for m's problem as: $V_m(b,q) = \max_{a \in A(b,q)} u_m(a)$ and note that $V_m(b,q) = u_m(a)$ for $a \in A^*(b,q)$ also holds.

Step 3: We show that given $A^*(b,q)$, M_1's best response $b^*(q)$ is well defined.

M_1's problem is to propose $b(q) \in \arg\max_{b \in W(q)} u_{M_1}(a^*)$ where $a^* \in A^*(b,q)$. For fixed $q \in PS$, we consider the following covering of PS: $\{E_{112}(q), E_{11m}(q), E_{12m}(q), E_{212}(q), E_{21m}(q), E_{22m}(q)\}$ and the following six problems:

$$\max_{b \in E_{112}(q) \cap PS} \{u_{M_1}(a_{112}(b,q))\} \text{ s.t. } u_m(a_{112}(b,q)) = V_m(b,q)$$

$$\max_{b \in E_{11m}(q) \cap PS} \{u_{M_1}(a_{11m}(b,q))\} \text{ s.t. } u_m(a_{11m}(b,q)) = V_m(b,q)$$

$$\max_{b \in E_{12m}(q) \cap PS} \{u_{M_1}(a_{12m}(b,q))\} \text{ s.t. } u_m(a_{12m}(b,q)) = V_m(b,q)$$

$$\max_{b \in E_{212}(q) \cap PS} \{u_{M_1}(a_{212}(b,q))\} \text{ s.t. } u_m(a_{212}(b,q)) = V_m(b,q)$$

$$\max_{b \in E_{21m}(q) \cap PS} \{u_{M_1}(a_{21m}(b,q))\} \text{ s.t. } u_m(a_{21m}(b,q)) = V_m(b,q)$$

$$\max_{b \in E_{22m}(q) \cap PS} \{u_{M_1}(a_{22m}(b,q))\} \text{ s.t. } u_m(a_{22m}(b,q)) = V_m(b,q)$$

We let $V_{ijk}^{M_1}(q)$ be the value function of the appropriate optimization problem above if a solution exists and $-\infty$ otherwise. The facts that $a_{ijk}(b,q)$ is continuous in $b \in E_{ijk}(q)$ (shown in step 2) and $u_{M_1}(x)$ is continuous imply that the objective functions in the above problems are continuous. From above we know that both sides of each constraint are continuous. Thus, the constraint set generated by the constraint is closed. So by $E_{ijk}(q) \cap PS$ compact the constraint set is compact. Moreover, $E_{112}(q) \cap E_{11m}(q) \cap E_{12m}(q) \cap E_{212}(q) \cap E_{21m}(q) \cap E_{22m}(q) \cap PS \neq \emptyset$ and for at least one of these sets a b exists that satisfies the constraint. Thus at least one of the problems has a non empty compact constraint set, and all problems have a compact constraint set. This implies that by the Theorem of the Maximum solutions to each of the programs for which the constraint set is non-empty exist. Denote these solution sets as $B_{ijk}^*(q)$. Let $a^*(b,q) := \arg\max_{a \in A^*(b,q)} u_{M_1}(a)$. Without loss of generality we assume that m will choose $a^*(b,q)$ - one of the best responses for m given (b, q).[6] This assumption implies that if $b \in E_{ijk}(q)$ and $E_{ijk}(q) \neq \emptyset$, then $u_m(x(b, q)) = V^{M_1}_{ijk}(q)$. Given this, M_1's optimal proposal is thus $b^*(q) \in \{\ B^*_{i'j'k'}(q)$:

$V^{M_1}_{i'j'k'}(q) \in \max\{V^{M_1}_{ijk}(q)\}_{ijk\in\{1,2\}\times(1,2)\times N}\}$. Since the number of $V^{M_1}_{ijk}(q)$'s is finite and M_1 is indifferent between points in a given $B^*_{ijk}(q)$, the transitivity of M_1's preferences implies that $b^*(q)$ exists. We denote the value function as $V_{M_1}(q) = u_{M_1}(a^*(b^*(q), q))$ (a fact used in the proof of the next lemma.)

Combining the steps yields the existence of a subgame perfect equilibrium in weakly undominated strategies.■

We do not reprove uniqueness here.

Lemma 4 Uniqueness: *For all $q \in PS$ the equilibrium is unique.*

Proof. The lemma is a special case of the main theorem of Austen-Smith (1987). An additional application of the Theorem of the Maximum shows that the equilibrium policy is itself a continuous function of the status quo.

Lemma 5 Continuity: *The equilibrium policy $x^*(q)$ is a continuous function, $x^* : PS \to PS$.*

Proof: By lemma 3 the policy exists, and by lemma 4 it is unique, thus $x^*(q)$ is a function on domain PS. To establish continuity note that by preferences continuous the sets $R_i(y)$ are continuous correspondences for $y \in X$. Thus the sets $E_{ijk}(q)$ are continuous correspondences. So by the Theorem of the Maximum and arguments made in the proof of lemma 3, $a_{ijk}(q)$ and $V_m(b, q)$ are continuous functions of q. Thus, the constraints in equations (1) through (6) are continuous correspondences of q. Again by the Theorem of the Maximum and previous arguments, $V^{M_1}_{i'j'k'}(q)$ are continuous in q. This and the continuity of the operator max implies that $V_{M_1}(q)$ is continuous in q. Since $V_m(b, q)$ and $V_{M_1}(q)$ denote the distance between m, and $x^*(q)$ and M_1 and $x^*(q)$ respectively the fact that these two distances are continuous in q implies that $x^*(q)$ is continuous in q as it is uniquely determined by these distances.■

---

[6] This assumption was shown to be neessary for existence in Banks and Gasmi (1987).

Lemma 6 Open set q-power: *Suppose preferences R_i are representable by a continuous utility function, $u_i(x)$, if $x(q)$ is continuous in q, and $\exists q' \in PS$ s.t. $q' P_i x(q')$. Then there exists a nonempty open set L_i over which i is not q-powerful.*

> **Proof**. Assume the hypotheses. By the continuity of $u(x)$, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t. $\forall q \in B(q', \delta_\varepsilon)$, we have $u_i(q') \in (u_i(q') - \varepsilon, u_i(q) + \varepsilon)$. By the continuity of $x(q)$ and the continuity of $u_i(x)$, the composition $u_i(x(q))$ is continuous in q. This implies that $\forall \varepsilon' > 0, \exists \delta'_{\varepsilon'} > 0$ s.t. $\forall q \in B(q', \delta'_{\varepsilon'})$, we have $u_i(x(q')) \in (u_i(x(q')) - \varepsilon', u_i(x(q)) + \varepsilon')$. By $x(q') \in P_i^{-1}(q')$, we have $u_i(x(q')) < u_i(q)$. This implies $\exists \varepsilon''$ s.t. $u_i(x(q')) < u_i(q) + \varepsilon''$. Selecting ε and ε' from above so that $\max\{\varepsilon, \varepsilon'\} < \frac{\varepsilon''}{2}$ implies that $\forall q \in B(q', \delta_m)$, we have $u_i(x(q)) < u_i(q)$ where $\delta_m = \min\{\delta_\varepsilon, \delta'_{\varepsilon'}\}$. Letting $B(q', \delta_m) = L_i$, we have constructed the desired non empty open set■
>
> Together, Lemma 6, the continuity of preferences and of $x(q)$, openness of $B(q', \delta^*)$, and $q' \in PS$, imply the existence of a nonempty L_i as stated in:

Corollary 1 *If $\exists q_i$ s.t. $q_i \, P_i \, x(q_i)$ then L_i has a non-empty interior.*

3 Propositions

Two geometric definitions are used. First, let $d(x, y)$ represent the Euclidean distance between points x and y. Second, let

$$\begin{aligned}
\mathcal{M}(x, y) &= \frac{1}{2}(x + y) \\
\mathcal{M}_a &= \mathcal{M}(x_m, x_{M_1}) \\
\mathcal{M}_b &= \mathcal{M}(x_{M_1}, x_{M_2}) \\
\mathcal{M}_c &= \mathcal{M}(x_m, x_{M_2})
\end{aligned}$$

In other words, $\mathcal{M}$ signifies a midpoints between a given pair of points, and the three cases of midpoints between pairs of ideal points will be of special interest.

The lemmas and some elementary geometry allow us to characterize the equilibrium path without solving for the complete equilibrium. The defense for this approach lies in the fact that a complete equilibrium characterization (even for the simple case of Proposition 1) requires tedious manipulation of high order polynomials that add no behavioral intuition. In contrast, specification of the equilibrium path illuminates behavior quite well. For example, the key insight of the proof of Proposition 1 is that, when the minority party proposer m behaves optimally, the majority-party member M_1 proposes a bill b that make him and M_2 equally attractive coalition partners to m. Next, m is indifferent between either of two amendments a, one of which benefits and thus includes M_1 in the coalition and the other of which attracts M_2's support. The geometry of an equilateral triangle is such that these conditions are met when $b = \mathcal{M}_a$, which elicits an optimal response $a = \mathcal{M}_b$. The proof formalizes this logic as illustrated in Figure 4.

Proposition 1 Equilateral triangle, central status quo: *Suppose ideal points* $\{x_i\}$ *and the status quo* q *satisfy the following conditions: (1)* $d(x_i, x_j) = d$ *constant* $\forall i, j \in N$*; and (2)* $d(q, x_i) = k$ *constant* $\forall i \in N$*. Then:*

(a) A unique equilibrium exists.

(b) $b^* = \mathcal{M}(x_{M_1}, x_{M_2}) = \mathcal{M}_b$*.*

(c) $a^*(b^*) = \mathcal{M}(x_m, x_{M_1}) = \mathcal{M}_a$*.*

(d) On the path, voting is as follows:

$$\begin{aligned} v_i^{*1} &= 1 \textit{ for } i \in \{M_1, m\} \\ v_{M_2}^{*1} &= 0 \\ v_i^{*2} &= 1 \textit{ for } i \in \{M_1, m\} \\ v_{M_2}^{*1} &= 0 \end{aligned}$$

(e) m *has q-power and* M *does not.*

Proof: (a) Follows immediately from Lemma 3 (existence) and Lemma 4 (uniqueness).

(c) Via backwards induction, we assume that (b) holds and illustrate that $a^*(\mathcal{M}_b) = \mathcal{M}_a$. From Lemmas 1 and 2 we know $a^*(\mathcal{M}_b) \in PS \cap W(q) \cap W(\mathcal{M}_b)$. $a^* \in W(\mathcal{M}_b)$ implies that $d(a^*, x_i) \leq d(\mathcal{M}_b, x_i)$, $\exists i \in M$. For the moment ignore the constraint $W(q)$. By the definition of preferences, m selects $C_m(\{a : d(a, x_i) \leq d(\mathcal{M}_b, x_i)\, \exists i \in M\})$. Letting $A \equiv \{a : d(a, x_i) \leq d(\mathcal{M}_b, x_i)\, \exists i \in M\}$, m's problem is equivalent to $\arg\min_{a \in A} d(a, x_m)$, which has solution $a^*(\mathcal{M}_b) \in \{\mathcal{M}_a, \mathcal{M}_c\}$. Note that these two points lie in $W(q)$. Resolving indifference in favor of $\mathcal{M}_a$ to resolve an open set problem yields the result.

(b) Given that m's best response to $b = \mathcal{M}_b$ is known to be $\mathcal{M}_a$ (part c), it is sufficient to show that no b' exists for which $a^*(b') P_{M_1} \mathcal{M}_a$. We proceed by contradiction. Assume $\exists b' \in PS$ s.t $u_{M_1}(a^*(b')) > u_{M_1}(\mathcal{M}_a)$. Since $\mathcal{M}_b$ is on the contract curve between m and M_1 this implies that $u_m(a^*(b')) < u_m(\mathcal{M}_a)$. Because $b' \neq \mathcal{M}_b$ either (i) $u_{M_1}(b') > u_{M_1}(\mathcal{M}_b)$ or (ii) $u_{M_2}(b') > u_{M_2}(\mathcal{M}_b)$.

-Assume (i). Since $\mathcal{M}_b$ is on the contract curve between M-party members, $u_{M_2}(b') \leq u_{M_2}(\mathcal{M}_b)$. But this implies, by transitivity, that $u_{M_2}(\mathcal{M}_a) \geq u_{M_2}(b')$. Because $u_m(a^*(b')) \geq u_m(b')$ (otherwise m would not propose $a^*(b')$), and $u_m(\mathcal{M}_a) > u_m(a^*(b'))$, transitivity implies that $u_m(\mathcal{M}_c) = u_m(\mathcal{M}_a) > u_m(b')$. This inequality and $u_{M_2}(b') \leq u_{M_2}(\mathcal{M}_b) = u_{M_2}(\mathcal{M}_c)$ imply that $\mathcal{M}_c \in W(b')$, because m and M_2 would vote for $\mathcal{M}_c$ over b'. Finally a contradiction in the optimality of $a^*(b')$ is obtained from $\mathcal{M}_c \in W(q)$ and $u_m(a^*(b')) < u_m(\mathcal{M}_a) = u_m(\mathcal{M}_c)$. The proposal $a^*(b')$ is dominated by $\mathcal{M}_c$, so (i) is false.

-Assume (ii). Since $\mathcal{M}_b$ is on the M-party contract curve, $u_{M_1}(b') \leq u_{M_1}(\mathcal{M}_b)$. But similarly, since $u_{M_1}(\mathcal{M}_b) = u_{M_1}(\mathcal{M}_a)$, and thus $u_{M_1}(b') \leq u_{M_1}(\mathcal{M}_a)$, and $u_m(a^*(b')) < u_m(\mathcal{M}_a)$, we have $\mathcal{M}_a \in W(b') \cap W(q)$. Thus $\mathcal{M}_a$ dominates $a^*(b')$ and we again have a contradiction. Since (i) and (ii) are exhaustive the result follows: $b^* = \mathcal{M}(x_{M_1}, x_{M_2}) = \mathcal{M}_b$.

(d) This follows immediately from Def. 2 (equilibrium).

(e) By (d), $x^* = \mathcal{M}_a$. Since $2k > d$, $d(x_i, x^*) = \frac{d}{2} < k = d(x_i, x^*)$ for $i \in \{M_1, m\}$. Therefore, M_1 and m have q-power. Since $q \in PS$, this implies $u_{M_2}(x^*) < u_{M_2}(q)$, so M_2 does not have q-power. Therefore, M does not have q-power.■

Proposition 2 extends the analysis of Proposition 1 to consider any status quo in the Pareto set. The logic used in Proposition 1 to characterize the equilibrium path depends on the status quo only to the extent that $W(q)$ contains $\mathcal{M}_a$, $\mathcal{M}_b$, and $\mathcal{M}_c$. For any q for which this occurs, equilibrium behavior is equivalent to the one for the centrally located q. When this condition is not satisfied, however, at least one of the proposers faces a constraint that is not present in instances of a centrally located status quo. Nonetheless, the indifference condition identified for M_1's proposal remains relevant.

Proposition 2 Equilateral triangle, Pareto status quo: *Suppose ideal points* x_i *and the status quo* q *satisfy the following conditions: (1)* $d(x_i, x_j) = d$ *constant* $\forall i, j \in N$*, and (2)* $q \in PS$*. Then:*

(a) A unique equilibrium exists.

(b) M_1 *and* m *vote in opposition to* M_2 *in the vote between* a *and* b*.*

(c) There exists a nonempty $C \subset PS$ *for which* $q \in C$ *implies that (b)-(d) of Proposition 1 hold.*

(d) $\forall i \in N$, $\exists L_i \subset PS$ *with a nonempty interior that contains* x_i *s.t.* $q \in L_i$ *implies* i *is not* q*-powerful.*

(e) No player has u-power.

Proof: (a) Follows immediately from Lemmas 3 (existence) and 4 (uniqueness).

(b) Clearly, $a^*(b) \in \arg\min_a\{d(a, x_m) \text{ s.t. } a \in W(q) \cap W(b^*(q))\}$. This implies $v_m(a^*, b^*) = 1$. It follows from m's problem, as defined in the proof of Lemma 3, that $a^*(b^*) I_{M_1} b^*(q)$. Because indifference is resolved in favor of the last proposer (now m), $v_{M_1}(a^*, b^*) = 1$. Because voting is sincere at the final node, and since $a^*, b^* \in PS$ by Lemma 2, $v_{M_2}(a^*, b^*) = 0$.

(c) In the proof of (b)-(d), q enters only through the requirement that Lemmas 1 and 2 must obtain. Therefore, if $\mathcal{M}_b \in W(q)$, $\mathcal{M}_a \in W(q)$, and $\mathcal{M}_c \in W(q)$ the equilibrium is as characterized in Proposition 1 (b)-(d). We therefore construct $C \subset PS$ such that the above conditions hold. The set C (depicted in Figure 5) satisfies these conditions.

(d) Since $x_i \in C$ and $x_i \, P_i \, \mathcal{M}_a \, \forall i \in N$, we have established that L_i is nonempty and contains x_i. This and corollary 1 imply that L_i has non-empty interior $\forall i \in N$.

(e) Follows immediately from (d).■

Proposition 3 extends the analysis of Proposition 1 to the case in which the triangle is not equilateral. Although the geometric techniques used above are no longer sufficient to characterize the equilibrium path, the underlying behavior is qualitatively similar.

Proposition 3 General triangle, Pareto status quo: *Suppose ideal points $\{x_i\}$ are three distinct points and $q \in PS$. Then:*

(a) A unique equilibrium exists.

(b) M_1 and m vote as a coalition against M_2 in the vote between a and b.

(c) $\forall i \in N \; \exists L_i \subset PS$ with a nonempty interior that contains x_i s.t. $q \in L_i$ implies i is not q-powerful.

(d) No player has u-power.

Proof: (a) Follows immediately from Lemmas 1 and 2.

(b) Follows from the proof of Proposition 2b which is not dependent on the triangle being equilateral.

(c) For all players, $x_i \in L_i$. Because $W(x_i) \cap_{j \neq i} P_j(x_i) \neq \emptyset$, the equilibrium outcome associated with any ideal point $x^*(x_i) \neq x_i$. By strict convexity, $x_i \, P_i \, x^*(x_i)$. Applying Corollary 1 yields the result.

(d) Follows immediately from (c).■

The final proposition characterizes the relationship between majority-party heterogeneity (as formulated by Aldrich and Rohde (1998)) and policy extremism (defined as the distance between the outcome and a central status quo). Formally, heterogeneity is $r \equiv d(x_{M_1}, x_{M_2})$. The variable r can be interpreted as a formalization of Rohde's condition for conditional party government.

A few other definitions clarify the analysis. Let $s = d(x_m, \mathcal{M}_a)$. This is the length of the segment connecting the minority party ideal point with the midpoint of majority-party members' ideal points. For isosceles triangles, it follows that the line segments $\overline{x_m, \mathcal{M}_b}$ and $\overline{x_{M_1}, x_{M_2}}$ are perpendicular, as shown in Figure 7.

We hold s fixed at 1 and let r go from 0 to $\frac{2\sqrt{3}}{3}$. For $r = 0$, all ideal points lie on a straight line, and the majority party is perfectly homogeneous. Intermediate cases of $s = 1$ and $r < \frac{2\sqrt{3}}{3} < 0$ satisfy the conditions of Proposition 3. The maximum instance of heterogeneity, $s = 1$ and $r = \frac{2\sqrt{3}}{3}$, meets the conditions of propositions 1 and 2.

It follows also that we can express ideal points as a function of r, denoted $x_i(r)$ for $i \in N$. Moreover, $x_m(r)$ may be held constant without loss of generality. We denote the center of the triangle, and location of the status quo, as $q \equiv c(r) \equiv \sum_{i \in N} \frac{1}{3} x_i \in PS$. A convenient property of $c(r)$ is that it is constant in r. Finally, let $x^*(r)$ denote the equilibrium outcomes associated with various parameters, i.e., the $x^*(c(r))$ induced by $x_i(r)$ for $i \in N$.

Proposition 4 Heterogeneity and individual utility: *Under the assumptions and parameterization given above, (i) the function* $U_i : [\frac{2\sqrt{3}}{3}, 0] \rightarrow \Re$*, defined by* $U_i(r) = u_i(x^*(r))$*, satisfies the following properties:*

(a) $U_i(\cdot)$ *is increasing on* $[\frac{2\sqrt{3}}{3}, 0]$ *for* $i \in M$ *and decreasing for* $i = m$*;*

(b) it has the graph approximated in Figure 8.

(c) individual gains $g_i(r) = U_i(r) - u_i(q)(r)$ *have the graph approximated in Figure 9.*

Proof: (a) $x^*(r^u) = \mathcal{M}_a$ for $r^u = \frac{2\sqrt{3}}{3}$ by Proposition 1. So,

$$U_i(\frac{2\sqrt{3}}{3}) = u_i(\mathcal{M}_a). \qquad (*)$$

Moreover, (a and b) $x_{M_1}(0) = x_{M_2}(0) = \mathcal{M}(x_{M_1}(0), x_{M_2}(0)) = \mathcal{M}_b$. $\mathcal{M}_b$ is the unique core point, and, from Austen-Smith (1987), this implies that $x^*(0) = \mathcal{M}_b$, so $U_i(0) = u_i(\mathcal{M}_b)$. The remainder of the proof is constructive and numeric. Following the logic of the proof of Proposition 1(b) we have $b^*(c(r)) = \mathcal{M}(x_{M_1}(r), x_{M_2}(r))$. By definition $a^*(b^*(c(r)))$ solves the program $\min_{x \in PS} d(x, x_m)$ s.t. $x \in W(c(r)) \cap W(b^*(c(r)))$. Figure 7 illustrates the numeric solution to this problem for a collection of problems parameterized by $r \in$

$[\frac{2\sqrt{3}}{3}, 0]$. The points in Figure 8 plot the corresponding graphs of U_i as a function of Rohde's homogeneity condition, r.

(c) Figure 9 is generated from figure 7 by plotting $U_i(r) - u_i(q)(r)$ on $r \in [\frac{2\sqrt{3}}{3}, 0]$.■

It can be shown that $x^*(r)$ is a continuous function of r. This suggests that the curve fit to the numeric solution (depicted in Figure 8) will closely approximate the true function.