CONNECTIONS BETWEEN $SL(2,\mathbb{R})$ AND TRUNCATED MOMENT PROBLEMS FOR THE UPPER-HALF PLANE

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Brief description of research project. Given a natural number $n \geq 1$ and a finite collection of real numbers $\beta \equiv \{\beta_{i,j}\}_{0 \leq i+j \leq 2n}$, the *truncated moment problem* (TMP) for β entails finding necessary and sufficient conditions on β for the existence of positive numbers $\rho_1, ..., \rho_r$ and points $(s_1, t_1), ..., (s_r, t_r) \in \mathbb{R}^2$ such that

$$(*) \qquad \beta_{i,j} = \sum_{h=1}^{r} \rho_h s_h^i t_h^j \ \text{ (all } i, j).$$

We call the sum $\mu \equiv \mu(\beta) := \sum_{h=1}^{r} \rho_h \delta_{(s_h,t_h)}$ a *representing measure* for β; here $\delta_{(s_h,t_h)}$ denotes the point mass at (s_h, t_h). Thus, when TMP is soluble, there is a correspondence between suitable β's and their associated μ's. Conversely, given μ one can always build a finite collection β by using (*) to define the numbers β_{ij} In short, every μ gives rise to a β, but not all β's have a representing measure μ.

Associated with each TMP is the symmetric matrix

$$M(n) \equiv M(n)(\beta) := (\beta_{i+k,j+\ell})_{0 \leq i+j \leq 2, 0 \leq k+\ell \leq 2},$$

of size $m \times m$, where $m := \frac{(n+1)(n+2)}{2}$; $M(n)$ is called the *moment matrix* of β. If β admits a representing measure μ, it is straightforward to prove that all eigenvalues of $M(n)$ are nonnegative; when that happens, we write $M(n) \geq 0$. Thus, a necessary condition for the solubility of TMP is $M(n) \geq 0$.

Consider now four real numbers a, b, c and d, and assume that $ad - bc = 1$. The *fractional linear transformation* $\varphi \equiv \varphi(a, b, c, d) : \mathbb{C} \longrightarrow \mathbb{C}$ given by $\varphi(z) := \frac{az+b}{cz+d}$, defines an automorphism of the upper-half plane $U := \{z \in \mathbb{C} : \text{Im } z > 0\}$. For instance, $\varphi(z) := \frac{z}{z+1}$ maps the horizontal line $L := \{t+i : t \in \mathbb{R}\}$ to the circle $C := \{(x, y) : (x-1)^2 + (y-\frac{1}{2})^2 = \frac{1}{4}\}$, and the points in the half plane above L to the points inside C.

We will study how the TMP changes when we let φ act on it. For instance, if we let $\varphi^*(\mu) := \mu \circ \varphi := \sum_{h=1}^{r} \rho_h \delta_{\varphi(s_h,t_h)}$, and if β and $\varphi^*(\beta)$ correspond to μ and $\varphi^*(\mu)$, how are the matrices $M(n)(\beta)$ and $M(n)(\varphi^*(\beta))$ related? Given a matrix $M(n)(\beta) \geq 0$, can we classify all other matrices K such that $K = M(n)(\varphi^*(\beta))$ for some automorsphism φ?

Research Project Goals. To unravel the precise connection between automorphisms of U and the structure of moment matrices. We will begin with $n = 1$ and $n = 2$, and discover as many features of the problem as possible, before attempting to generalize to bigger values of n. Along the way we will discover the rudiments of the theory of classical moment problems.

Prerequisites. Single variable calculus, linear algebra, some basic knowledge of inner product spaces, and some basic knowledge of complex numbers. Many of our calculations will be done using *Mathematica*, so some familiarity with symbolic manipulation is desirable, although not essential.

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