Automorphisms and Distinguishing Labelings of Graphs

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Outline

1. Basics of Graphs and Their Automorphisms
2. Distinguishing Labeling and Number
3. Number of Distinguishing Labelings on Paths
4. Automorphism Groups

Definitions

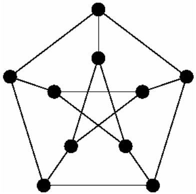
Definition

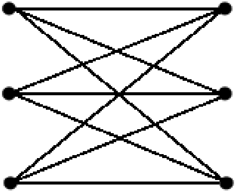
A graph *G is an ordered pair (V, E), where V is a nonempty finite set and E is a set of 2-element subsets of V. The members of V and E are called* vertices *and* edges*, respectively.*

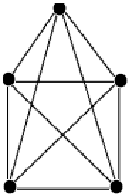
Definition

If two edges have a vertex in common, we say that they are adjacent*. If $e = \{u, v\}$ is an edge, we say that u and v are* adjacent*.*

Examples of Graphs





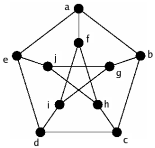


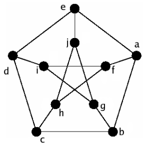
Definitions

Definition

Consider graphs G *and* H *and a bijection* $f : V(G) \to V(H)$*. We call* f *an* isomorphism *from* G *to* H *when* $uv \in E(G)$ *if and only if* $f(u)f(v) \in E(H)$*. When such an* f *exists, we say that* G *is* isomorphic *to* H *and write* $G \simeq H$*.*

An automorphism *of a graph is an isomorphism from the graph to itself.*





Definitions

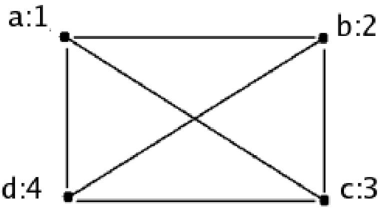
Definition

An n-labeling *of a graph is a function* $f : V \rightarrow \{1, 2, ..., n\}$.

A labeling of a graph is distinguishing *if no nontrivial automorphism of the graph preserves it.*

The distinguishing number, *D, of a graph is the smallest n such that the graph has a distinguishing n-labeling.*

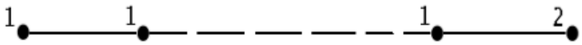
A Distinguishing Label Example: K_4



Distinguishing Numbers

Theorem

The distinguishing number of a path graph is 2.

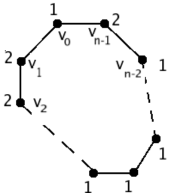


Proof.

Label one end vertex 1, the other end vertex 2 and finish by labeling all the other vertices 1. Since the only nonidentity automorphism of a path graph swaps the end vertices, this is a distinguishing labeling. □

Theorem

The distinguishing number of an n-cycle for $n > 5$ is 2.

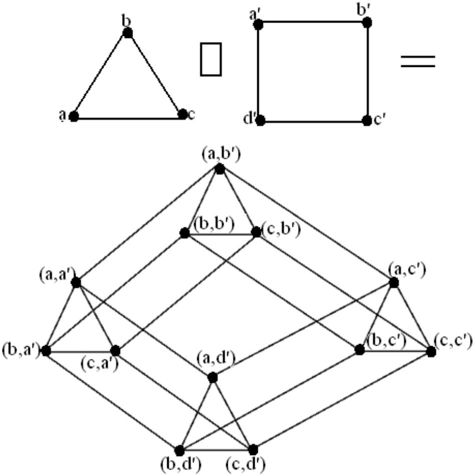


Proof.

Pick a vertex, call it v_0. Label that vertex 1 and its neighbors v_1 and v_{n-1} both 2. Next, label vertex v_2 with a 2 and v_{n-2} 1. Finally, label the remaining vertices 1. □

Definition

Given graphs G and H with disjoint vertex sets, the *Cartesian product* $G \square H$ has vertex set $\{(u, v) : u \in V(G), v \in V(H)\}$ where vertices (u, v) and (u', v') are adjacent when $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$.


b
a
c
a′
b′
d′
c′
=
(a,b′)
(b,b′)
(c,b′)
(a,a′)
(a,c′)
(b,a′)
(c,a′)
(a,d′)
(b,c′)
(c,c′)
(b,d′)
(c,d′)

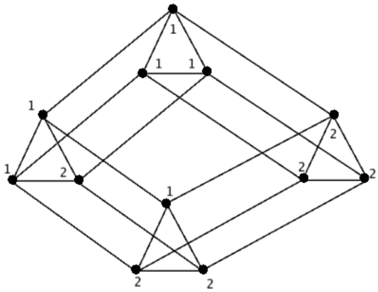
Theorem

If G and H are not isomorphic, then $D(G \Box H) \leq max\{D(G), D(H)\}$.

Proof.

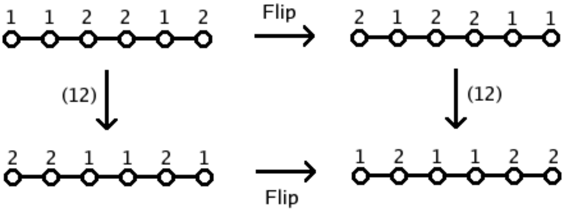
Treat every copy of G as a single vertex of H and give H a distinguishing labeling using $D(H)$ labels, which leaves each copy of G with the same labels within itself. This labeling prevents an automorpism of $(G \Box H)$ that moves the copies of G from preserving adjacency. Now give one of the copies of G a distinguishing labeling using $D(G)$ labels. Then any automorphism of $(G \Box H)$ that moves the vertices of any copy of G, and thus all copies of G, must disrupt the labeling of this copy. □

An example where $D(G\Box H) < min\{D(G), D(H)\}$ is $C_3\Box C_4$, where $D(C_3) = 3$ and $D(C_4) = 3$ but $D(C_3\Box C_4) = 2$.



Definition

Let $DL_c(G)$ denote the set of distinguishing labelings of a graph G using c colors where two distinguishing labelings of G are considered equivalent if there exists an automorphism taking one labeling to the other labeling using automorphisms of G and permuting the colors on G.



Case I: Labelings on P_n which start and end with a 1.

$$\underline{1}|\ \underline{a}\ \underline{b}\ \underline{c}\ \ldots\ \underline{d}\ \underline{e}\ \underline{f}\ |\underline{1}$$

There are 2^{n-2} ways of labeling the remaining $(n-2)$-elements.

$$\underline{1}|\ \underline{1}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{1}\ |\underline{1}$$

Palindromes are not distinguishing.

flip :

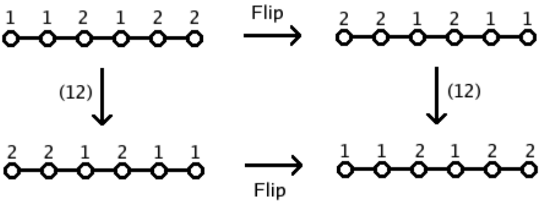
$$\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}\ \underline{e}\ \underline{f} \longmapsto \underline{f}\ \underline{e}\ \underline{d}\ \underline{c}\ \underline{b}\ \underline{a} \qquad \underline{a}\ \underline{b}\ \underline{c}\ \underline{d}\ \underline{e} \longmapsto \underline{a}\ \underline{b}\ \underline{c}\ \underline{d}\ \underline{e}$$

$$\frac{2^{n-2} - 2^{(n-2)/2}}{2}$$

Case II: Labelings on P_n which start with a 1 and end with a 2

1| a b c ... d e f |2

There are 2^{n-2} ways of labeling the remaining $(n-2)$-elements.



$$\frac{2^{n-2} - 2^{(n-2)/2}}{2} + 2^{(n-2)/2} \qquad \frac{2^{n-2}}{2}$$

n even

$$| DL_2(P_n) | = [\frac{2^{n-2} - 2^{(n-2)/2}}{2}] + [\frac{2^{n-2} - 2^{(n-2)/2}}{2} + 2^{(n-2)/2}] = 2^{n-2}$$

n odd

$$| DL_2(P_n) | = [\frac{2^{n-2} - 2^{(n-1)/2}}{2}] + [\frac{2^{n-2}}{2}] = 2^{n-2} - 2^{(n-3)/2}$$

The set of all automorphisms of a graph forms a group under composition. We studied these groups and determined the automorphism group of several graphs.

Theorem

The distinguishing number of a graph is always less than or equal to the number of automorphisms of the graph.

Proof.

Consider a graph G. For every automorphism, we only need for one vertex that is mapped to another vertex under the automorphism to be a different color. Sometimes we don't even need a new color, so we won't add as many as the number of automorphisms. □

We now move onto the automorphism groups of several different graphs, but we must define a few types of graphs along the way.

Definition

When the graphs G and H have disjoint vertex sets, their *union* $G \cup H$ is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

The Wreath Product is a special case of the semi-direct product. When working in $GwrS_n$, we will have an n+1 tuple, with n components being elements from G, and the last element being a permutation.

Example

$$(g_1, g_2, ..., g_n, \alpha), (g\prime_1, ..., g\prime_n, \beta)$$

To multiply, $(g\prime_1, ..., g\prime_n, \beta)(g_1, ..., g_n, \alpha)$ we permute the right n+1 tuples' nonpermutation elements as beta says (i.e. (12) means the 1st goes to second and vice versa) and then multiply component wise.

Theorem

Given connected graphs G and H where G is not isomorphic to H, and $Aut(G) \simeq C$, $Aut(H) \simeq D$, *then* $Aut(G \bigcup H) \simeq C \times D$.

Proof.

When G and H are not isomorphic, there are no automorphisms that switch between components. Thus each automorphism is an automorphism of the graph G paired with an automorphism of the graph H. Thus, we can define a homomorphism $f : Aut(G \bigcup H) \rightarrow C \times D$. □

Notice that we can extend this theorem so that if we have *n* nonisomorphic connected graphs, the automorphism group of their union is just the direct product of all of their automorphism groups. Thus, we know

Theorem

Given graphs $G_1, G_2, ..., G_n$ *where* G_i *is not isomorphic to* G_j *for any i,j, and if* $Aut(G_i) \simeq D_i$*, then*
$Aut(G_1 \bigcup ... \bigcup G_n) \simeq D_1 \times ... \times D_n$

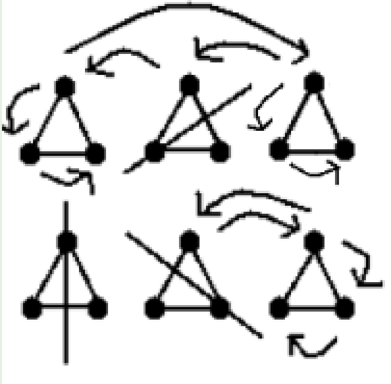
There is also a different case to consider, when there are multiple copies of the same connected graph in the union.

Theorem

Given graphs $G_1, G_2, ..., G_n$ *and where* $G_i \simeq G_j$ *for all* i, j, $Aut(G_1 \bigcup G_2 \bigcup ... \bigcup G_n) \simeq DwrS_n$, *where* $Aut(G) \simeq D$.

Example

Consider the following example of 2 $3K_3$s and automorphisms acting on them.



$$a \to i, b \to g$$
$$c \to h, d \to c$$
$$e \to a, f \to b$$
$$g \to e, h \to d$$
$$i \to f$$

Under T_l:

$$a \to a, b \to c$$
$$c \to b, d \to h$$
$$e \to i, f \to g$$
$$g \to f, h \to e$$
$$i \to d$$

Composing these two, we obtain:

$$a \to d, b \to f$$

$$c \to e, d \to b$$

$$e \to a, f \to c$$

$$g \to i, h \to h$$

$$i \to g$$

When multiplying the wreath product we obtain

$$(f_1, f_3, r_1, (23))(r_2, f_2, r_2, (132)) =$$

$$(f_1 r_2, f_3 r_2, r_1 f_2, (12))$$

Notice, $f_1 r_2 = f_2$, $f_3 r_2 = f_1, r_1 f_2 = f_3$. Thus, we have $(f_2, f_1, f_3, (12))$. This directly corresponds to our movement of letters upon checking.

Theorem

Let G and H be nonisomorphic graphs. Also let $Aut(G) \simeq C$ and $Aut(H) \simeq D$. Then $Aut(G \square H) \simeq C \times D$.

Proof.

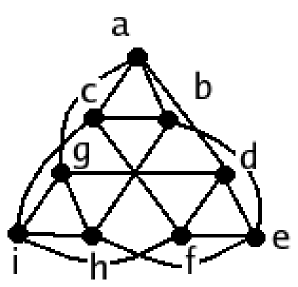
Every automorphism of $G \square H$ can be generated by applying an automorphism to the copies of G within $G \square H$ and then pairing it with an automorphism of H. Thus every element can be thought of as an ordered pair in $Aut(G) \times Aut(H)$. You can define a function that is a homomorphism between $Aut(G \square H)$ and $Aut(G) \times Aut(H)$. □

Theorem

Let G and H be isomorphic graphs where $Aut(G) \simeq C$*. Then* $Aut(G \square H) \simeq C wr S_2$*.*

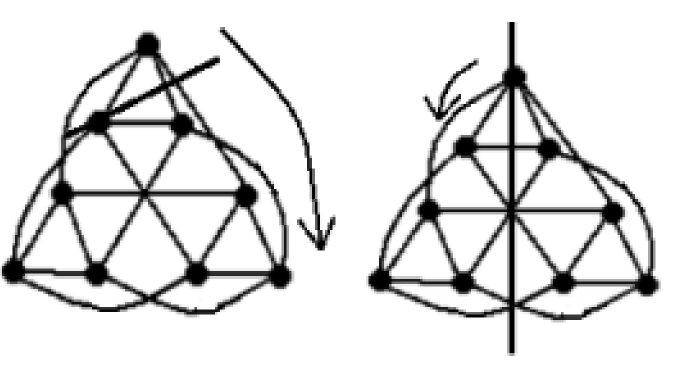
Example

Here is an example of how to apply the wreath product to $Aut(G \square G)$. Consider the following graph:



Example

Let us compose T_j and T_i, obtaining $T_j \circ T_i$. Each automorphism switches inside and outside, and applies an automorphism to each component, like this:



The left graph is the first automorphism applied, and it is composed with the other. Thus, here are the movements.

$$a \rightarrow e, b \rightarrow h$$

$$c \rightarrow b, d \rightarrow d$$

$$e \rightarrow g, f \rightarrow a$$

$$g \rightarrow f, h \rightarrow i, i \rightarrow c$$

We then apply the second automorphism:

$$a \rightarrow c, b \rightarrow a$$

$$c \rightarrow b, d \rightarrow i$$

$$e \rightarrow g, f \rightarrow h$$

$$g \rightarrow f, h \rightarrow d$$

$$i \rightarrow e$$

Composing these two automorphisms, we find:

$$a \to g, b \to d$$

$$c \to a, d \to i$$

$$e \to f, f \to c$$

$$g \to h, h \to e$$

$$i \to b$$

Let us now ask what our multiplication in the wreath product would yield. We should multiply $(r_2, f_1, ())(f_2, r_1, (12)) = (r_2 f_2, f_1 r_1, (12))$. Notice that $r_2 f_2 = f_1$ and $f_1 r_1 = f_3$. Thus we have $(f_1, f_3, (12))$, which corresponds to the overall effect of the automorphism and tells us that we move points from the inner graph to the outer graph.

Thanks to our mentors Aba and Michael!