Matrix Representations of Path Algebras

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Ring

Definition. *A* **ring** R *is a set together with two binary operations addition and multiplication that satisfy the following:*

1. $\forall a, b, c \in R, (a+b)+c = a+(b+c)$

2. $\exists 0 \in R$ *such that* $\forall a \in R, a+0 = 0+a = a$

3. $\forall a \in R, \exists -a \in R$ *such that* $a+(-a) = (-a)+a = 0$

4. $\forall a, b \in R, a+b = b+a$

5. $\forall a, b, c \in R, a(bc) = (ab)c$

6. $\forall a, b, c \in R, (a+b)c = ac+bc$ *and* $a(b+c) = ab+ac$

Examples of Rings

- $\mathbb{Z}$, the integers

- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$

- The set of all $n \times n$ matrices, $M_n(R)$, where R is a ring

- The set of all upper triangular $n \times n$ matrices with entries from a ring R

Homomorphism and Isomorphism

Definition. *Let* R *and* S *be two rings and let* f *be a map from* R *to* S. $f : R \longrightarrow S$ *is a* **ring homomorphism** *if* $\forall a, b \in R$, $f(a+b) = f(a) + f(b)$ *and* $f(ab) = f(a)f(b)$.

Definition. *A ring homomorphism which is one-to-one and onto is called a* **ring isomorphism**.

Quivers

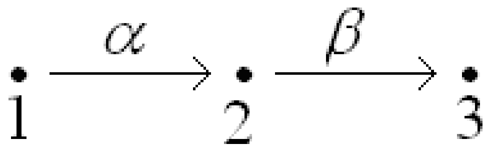
Definition. *A* **quiver** *is a directed graph* $Q = (Q_0, Q_1, s, e)$ *where* Q_0 *is the set of vertices,* Q_1 *is the set of arrows, and* $s : Q_1 \longrightarrow Q_0$ *and* $e : Q_1 \longrightarrow Q_0$ *are maps.*

Given an arrow $\alpha \in Q_1$ *with* $\alpha : i \longrightarrow j$ *for* $i, j \in Q_0$. *Then* s *and* e *are defined s.t.* $s(\alpha) = i$ *and* $e(\alpha) = j$, *i.e.* $s(\alpha)$ *is the vertex where* α *starts and* $e(\alpha)$ *is the vertex where* α *terminates.*

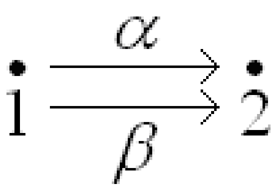
Note: A quiver is said to be finite provided Q_0 and Q_1 are both finite sets.

Examples of Quivers

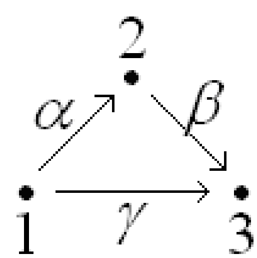
$Q_1 =$



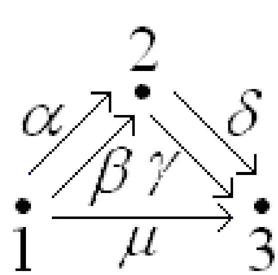
$Q_2 =$



$Q_3 =$



$Q_4 =$



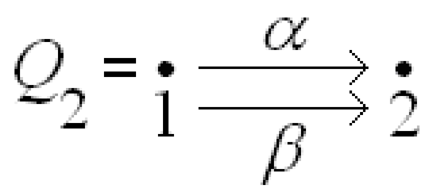
Paths

Definition. *A* **path** *in a quiver* Q *is either an ordered sequence of arrows* $p = \alpha_1 \cdots \alpha_{n-1}\alpha_n$ *with* $e(\alpha_t) = s(\alpha_{t+1}) for 1 \leq t < n$ *or the symbol* e_i *for* $i \in Q_0$.

The paths e_i, $i \in Q_0$ *are called the* **trivial paths**. *We define* $s(e_i) = e(e_i) = i$, *and we say* e_i *has length zero. For a nontrivial path* $p = \alpha_1 \cdots \alpha_{n-1}\alpha_n$, $s(p) = s(\alpha_1)$ *and* $e(p) = e(\alpha_n)$, *and the length of* p *is n.*

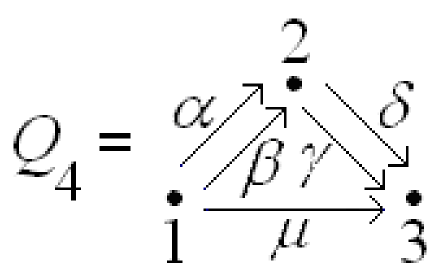
Examples of Paths

Let Q_2 be the quiver below.



e_1, e_2, α, and β are the paths in Q_2.

Let Q_4 be the quiver below.



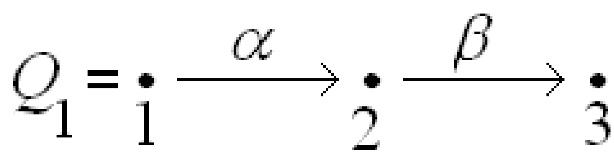
e_1, e_3, α, γ, $\alpha\delta$, and $\beta\gamma$ are examples of paths in Q_4.

Path Algebras

Definition. *Let k be a field. The* **path algebra** *kQ of the quiver Q is defined to be the k-vector space with basis the set of all paths in Q. The product of two paths is taken to be composition if it exists and zero otherwise.*

Example of path multiplication

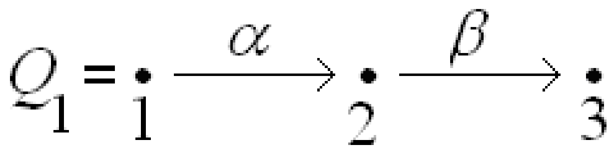
Consider the quiver below:



then $(e_1)(\alpha) = \alpha$, but $(\alpha)(e_1) = 0$. Also $(\beta)(\alpha) = 0$, but $(\alpha)(\beta) = \alpha\beta$.

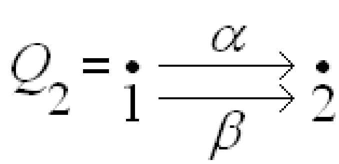
Examples of Path Algebras

1)Let k be a field and Q_1 be the quiver below.



Then $\{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}$ is a k basis for the path algebra kQ_1 over k.

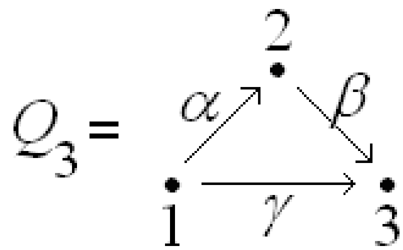
2)Let k be a field and Q_2 be the quiver below.



Then $\{e_1, e_2, \alpha, \beta\}$ is a k basis for the path algebra kQ_2 over k.

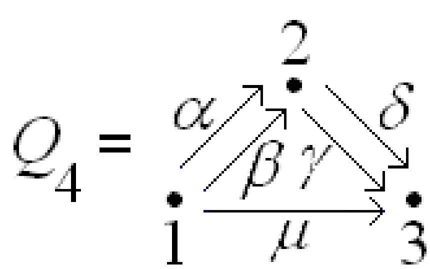
Examples of Path Algebras (cont.)

3)Let k be a field and Q_3 be the quiver below.



Then $\{e_1, e_2, e_3, \alpha, \beta, \gamma, \alpha\beta\}$ is a k basis for the path algebra kQ_3 over k.

4)Let k be a field and Q_4 be the quiver below.



Then $\{e_1, e_2, e_3, \alpha, \beta, \gamma, \delta, \mu, \alpha\delta, \alpha\gamma, \beta\delta, \beta\gamma\}$ is a k basis for the path algebra kQ_4 over k.

Adjacency Matrices

Definition. *For a quiver with n vertices, the* **adjacency matrix** A_Q *is an $n \times n$ matrix such that the $(i,j)^{th}$ entry is the number of arrows from i to j, where i and j are vertices.*

Theorem. *In A_Q^n, the $(i,j)^{th}$ entry is the number of paths of length n between vertices i and j.*

Proof. See Theorem 12.168 from Graph Theory Workshop Notes, Summer 2007 edition.

□

Theorem. *Let Q be a quiver with n vertices, if A_Q^n contains a nonzero entry, then the path algebra kQ is infinite.*

Proof. From the above theorem, we know that A_Q^n is the number of paths of length n in Q. If Q has no oriented cycles then the maximum path length between any two vertices is $n-1$. Therefore, if there is a nonzero element in A_Q^n there must be an oriented cycle. This will make the path algebra infinite dimensional. □

Matrix Representation of Path Algebras

Let k be a field and Q_1 the quiver below.

$$Q_1 = \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet}$$

Theorem. *$kQ_1 \cong M$ where M is the set of upper triangular 3×3 matrices with entries in k.*

Proof: Let $f : kQ_1 \longrightarrow M$ such that

$$a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha + a_5\beta + a_6\alpha\beta \longmapsto \begin{bmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{bmatrix}$$

Let $x = a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha + a_5\beta + a_6\alpha\beta$ and $y = b_1e_1 + b_2e_2 + b_3e_3 + b_4\alpha + b_5\beta + b_6\alpha\beta$.

- <u>f is a homomorphism</u>

$$f(x)+f(y) = \begin{bmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_4 & b_6 \\ 0 & b_2 & b_5 \\ 0 & 0 & b_3 \end{bmatrix} =$$

$$\begin{bmatrix} a_1+b_1 & a_4+b_4 & a_6+b_6 \\ 0 & a_2+b_2 & a_5+b_5 \\ 0 & 0 & a_3+b_3 \end{bmatrix} = f(x+y).$$

$$f(x)f(y) = \begin{bmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_4 & b_6 \\ 0 & b_2 & b_5 \\ 0 & 0 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1b_1 & a_1b_4+a_4b_2 & a_1b_6+a_4b_5+a_6b_3 \\ 0 & a_2b_2 & a_2b_5+a_5b_3 \\ 0 & 0 & a_3b_3 \end{bmatrix} = f(xy).$$

- <u>f is one-to-one</u>

Let $x, y \in kQ_1$ such that $f(x) = f(y)$. Thus,

$$\begin{bmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} b_1 & b_4 & b_6 \\ 0 & b_2 & b_5 \\ 0 & 0 & b_3 \end{bmatrix}.$$

Thus $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$, $a_5 = b_5$, and $a_6 = b_6$.

- <u>f is onto</u>

  Let $s \in M$ be arbitrary.

  Then $s = \begin{bmatrix} s_1 & s_4 & s_6 \\ 0 & s_2 & s_5 \\ 0 & 0 & s_3 \end{bmatrix}$ where $s_i \in k \ \forall \ i$.

  Every element of kQ_1 is of the form $a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha + a_5\beta + a_6\alpha\beta$ where $a_i \in k$. So there is an $a \in kQ_1 : a = s_1e_1 + s_2e_2 + s_3e_3 + s_4\alpha + s_5\beta + s_6\alpha\beta$ and $f(a) = s$. Therefore f is onto. $\square$

Theorem. *The quiver of Dynkin Diagram form A_n has path algebra isomorphic to M, the $n \times n$ upper triangular matrices with entries in k.*

$$Q_n = \underset{1}{\bullet} \xrightarrow{\ \alpha\ } \underset{2}{\bullet} \cdots \underset{n-1}{\bullet} \xrightarrow{\ \alpha_n\ } \underset{n}{\bullet}$$

Proof. Let Q_n be the quiver of Dynkin Diagram form A_n. Thus, Q_n has $n(n-1)/2$ non-trivial paths. Therefore Q_n has a basis with $n + n(n-1)/2 = n(n+1)/2$ elements. Let $f : kQ_n \longrightarrow M$ such that

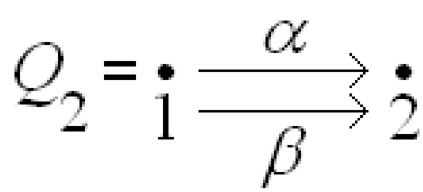
$$a_1e_1 + \cdots + a_ne_n + a_{n+1}p_1 + \cdots + a_{n(n+1)/2}p_{n(n-1)/2} \longmapsto \begin{bmatrix} a_1 & \cdots & a_{n(n+1)/2} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix}.$$

Proving that f is a homomorphism and that it is one-to-one and onto is done in a similar way to that of the previous theorem.

□

Matrix Representations cont.

Let k be a field and Q_2 the quiver below.



Let $M = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$.

Consider $f : kQ_2 \longrightarrow M$ such that

$x = a_1 e_1 + a_2 e_2 + a_3\alpha + a_4\beta \mapsto \begin{bmatrix} a_1 & (a_3, a_4) \\ 0 & a_2 \end{bmatrix}$

Then it can be shown that f is an isomorphism.

Matrix Representations cont.

Theorem. *Let k be a field and Q be the quiver with two vertices, labeled 1 and 2, and n arrows from vertex 1 to vertex 2.*
Let $M = \begin{bmatrix} k & k^n \\ 0 & k \end{bmatrix}$. Then, $kQ \cong M$.

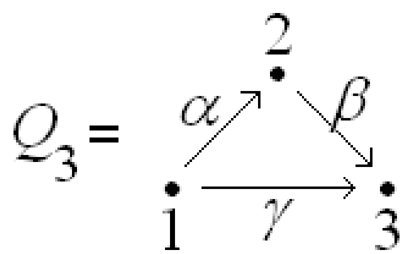
Proof: Let $f : kQ \longrightarrow M$ such that

$$a_1e_1+a_2e_2+a_3\alpha_1+a_4\alpha_2+\cdots+a_{n+2}\alpha_n \longmapsto \begin{bmatrix} a_1 & (a_3, .., a_{n+2}) \\ 0 & a_2 \end{bmatrix}$$

It can be shown that f is a ring isomorphism. $\square$

Matrix Representations cont.

Let k be a field and Q_3 the quiver below.



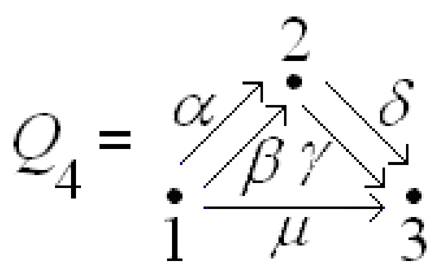
Let $M = \begin{bmatrix} k & k & k^2 \\ 0 & k & k \\ 0 & 0 & k \end{bmatrix}$. Consider $f : kQ_3 \longrightarrow M$ such that

$x = a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha + a_5\beta + a_6\gamma + a_7\alpha\beta$

$\mapsto \begin{bmatrix} a_1 & a_4 & (a_6, a_7) \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{bmatrix}$. Then it can be shown that $kQ_3 \cong M$.

Matrix Representation

Let k be a field and Q_4 the quiver below.



Let $M = \begin{bmatrix} k & k^5 & k^5 \\ 0 & k & k^5 \\ 0 & 0 & k \end{bmatrix}$. Consider $f : kQ_4 \longrightarrow M$ such that $a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha + a_5\beta + a_6\gamma + a_7\delta + a_8\alpha\gamma + a_9\alpha\delta + a_{10}\beta\gamma + a_{11}\beta\delta + a_{12}\mu \longmapsto$

$$\begin{bmatrix} a_1 & (a_4, a_4, a_5, a_5, 0) & (a_8, a_9, a_{10}, a_{11}, a_{12}) \\ 0 & a_2 & (a_6, a_7, a_6, a_7, 0) \\ 0 & 0 & a_3 \end{bmatrix}.$$

Theorem. *f is a one-to-one ring homomorphism.*

Proof:

- f is a homomorphism

$$f(x) + f(y) =$$

$$\begin{bmatrix} a_1 & (a_4, a_4, a_5, a_5, 0) & (a_8, a_9, a_{10}, a_{11}, a_{12}) \\ 0 & a_2 & (a_6, a_7, a_6, a_7, 0) \\ 0 & 0 & a_3 \end{bmatrix}$$

$$+ \begin{bmatrix} b_1 & (b_4, b_4, b_5, b_5, 0) & (b_8, b_9, b_{10}, b_{11}, b_{12}) \\ 0 & b_2 & (b_6, b_7, b_6, b_7, 0) \\ 0 & 0 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_1 & c_{1,2} & c_{1,3} \\ 0 & a_2 + b_2 & c_{2,3} \\ 0 & 0 & a_3 + b_3 \end{bmatrix} = f(x + y),$$

where

$$c_{1,2} = (a_4 + b_4, a_4 + b_4, a_5 + b_5, a_5 + b_5, 0),$$

$$c_{1,3} = (a_8 + b_8, a_9 + b_9, a_{10} + b_{10}, a_{11} + b_{11}, a_{12} + b_{12}),$$

$$c_{2,3} = (a_6 + b_6, a_7 + b_7, a_6 + b_6, a_7 + b_7, 0).$$

$$f(x)f(y) = \begin{bmatrix} a_1 & (a_4, a_4, a_5, a_5, 0) & (a_8, a_9, a_{10}, a_{11}, a_{12}) \\ 0 & a_2 & (a_6, a_7, a_6, a_7, 0) \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_1 & (b_4, b_4, b_5, b_5, 0) & (b_8, b_9, b_{10}, b_{11}, b_{12}) \\ 0 & b_2 & (b_6, b_7, b_6, b_7, 0) \\ 0 & 0 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1b_1 & c_{1,2} & c_{1,3} \\ 0 & a_2b_2 & c_{2,3} \\ 0 & 0 & a_3b_3 \end{bmatrix} = f(xy), \text{ where}$$

$c_{1,2} = (a_1b_4+a_4b_2, a_1b_4+a_4b_2, a_1b_5+a_5b_2, a_1b_5+a_5b_2, 0)$,

$c_{1,3} = (a_1b_8+a_4b_6+a_8b_3, a_1b_9+a_4b_7+a_9b_3, a_1b_{10}+a_5b_6+ a_{10}b_3, a_1b_{11} + a_5b_7 + a_{11}b_3, a_1b_{12} + a_{12}b_3)$,

$c_{2,3} = (a_2b_6+a_6b_3, a_2b_7+a_7b_3, a_2b_6+a_6b_3, a_2b_7+a_7b_3, 0)$.

- $\underline{f \text{ is one-to-one}}$ Let $x, y \in kQ_4$ such that $f(x) = f(y)$. Let $x = a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha + a_5\beta + a_6\gamma + a_7\delta + a_8\alpha\gamma + a_9\alpha\delta + a_{10}\beta\gamma + a_{11}\beta\delta + a_{12}\mu$ and $y = b_1e_1 + b_2e_2 + b_3e_3 + b_4\alpha + b_5\beta + b_6\gamma + b_7\delta + b_8\alpha\gamma + b_9\alpha\delta + b_{10}\beta\gamma + b_{11}\beta\delta + b_{12}\mu$. Since $f(x) = f(y)$, we have that

  $$\begin{bmatrix} a_1 & (a_4, a_4, a_5, a_5, 0) & (a_8, a_9, a_{10}, a_{11}, a_{12}) \\ 0 & a_2 & (a_6, a_7, a_6, a_7, 0) \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} b_1 & (b_4, b_4, b_5, b_5, 0) & (b_8, b_9, b_{10}, b_{11}, b_{12}) \\ 0 & b_2 & (b_6, b_7, b_6, b_7, 0) \\ 0 & 0 & b_3 \end{bmatrix}$$

  Thus $a_i = b_i$, $\forall i$

  $\square$

The above shows that kQ_4 is isomorphic to a subring of M, and it can easily be shown that this subring is isomorphic to $\begin{bmatrix} k & k^2 & k^5 \\ 0 & k & k^2 \\ 0 & 0 & k \end{bmatrix}$.

Future Work

For our future research, we would like to explore and understand the connection between the concrete method of finding matrix representations for path algebras that we have discovered and the following theorem.

Theorem. *Let $R = kQ$ for a quiver Q where Q is finite and has no oriented cycles. Then R is isomorphic to the following matrix:*

$$\begin{bmatrix} K & {}_1\tilde{K}_2 & \dots & {}_1\tilde{K}_n \\ & K & \dots & {}_2\tilde{K}_n \\ & & \ddots & \vdots \\ 0 & & & K \end{bmatrix}$$

where ${}_i\tilde{K}_j = \oplus_\gamma \gamma K$ and γ runs through all oriented paths from i to j. The multiplication is given by $K - K$-bilinear maps $c_{ijt} : {}_i\tilde{K}_j \otimes {}_j\tilde{K}_t \longrightarrow {}_i\tilde{K}_t$ induced by the multiplication in R.

Thank You!!!