On the graph $\Gamma(\mathbb{Z}_n)$

Jen Czachura, Eyob Meles, Jill Wiebke

Summer Research Experience for Undergraduates
July 25, 2007
Advisor: Professor D.D. Anderson
Mentor: Katie Reif

Outline

1. Definitions
   - Commutative Rings
   - Graph Theory Definitions
2. Main Results
   - Connectivity and Diameter
   - Clique and Chromatic Number
3. The Counterexample

Outline

1. Definitions
   - Commutative Rings
   - Graph Theory Definitions
2. Main Results
   - Connectivity and Diameter
   - Clique and Chromatic Number
3. The Counterexample

Definitions

Commutative Ring

Definition

A ***commutative ring*** R is a nonempty set with two binary operations "$+$" and "$\times$" that satisfies the following:

- $a + b = b + a$
- $(a + b) + c = a + (b + c)$
- $\exists\, 0 \in R$ with $a + 0 = a = 0 + a$ for $a \in R$
- $\exists -a \in R$ with $a + (-a) = 0 = -a + a$
- $ab = ba$
- $(ab)c = a(bc)$
- $\exists\, 1 \in R$ with $1a = a = a * 1$
- $a(b + c) = ab + ac$

Definitions
Commutative Ring

Examples of commutative rings include:

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_n$
- Polynomial rings $R[x] = \{a_0 + a_1x + ... + a_nx^n | a_i \in R\}$
- Direct products $R \times S = \{(r, s) | r \in R, s \in S\}$, $(r, s) + (r', s') = (r + r', s + s'), (r, s)(r', s') = (rr', ss')$

Definitions

Let R be a commutative ring. An element $r \in R$ is a ***zero divisor*** if there exists $0 \neq s \in R$ with $rs = 0$.

- 0 is always a zero divisor
- $Z(R)$ denotes the set of zero divisors of R
- $Z(R)^* = Z(R) - \{0\}$ is the set of proper zero divisors
- $Z(R)^* = \phi$ iff $Z(R) = \{0\}$ iff R is an integral domain

Definitions

Let R be a commutative ring with an identity. A nonempty subset $I \subseteq R$ is an ***ideal*** if the following hold:

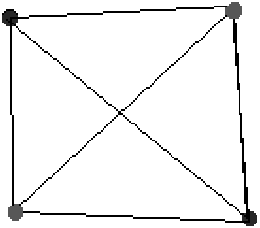
- $a, b \in I \implies a + b \in I$
- for $r \in R$ and $i \in I$, $ri \in I$

Outline

1. Definitions
   - Commutative Rings
   - Graph Theory Definitions
2. Main Results
   - Connectivity and Diameter
   - Clique and Chromatic Number
3. The Counterexample

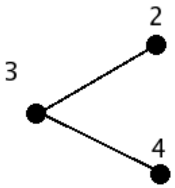
Definitions

- A ***graph*** G is a set of vertices connected by edges.
- A ***complete graph*** has every vertex adjacent to every other vertex.



Definitions

- The ***zero divisor graph*** of R is $\Gamma(R)$ where vertices of $\Gamma(R)$ are the non-zero zero divisors $Z(R)^*$. Two vertices $x, y \in \Gamma(R)$ are adjacent iff $xy = 0$.



Zero divisor graph of $\mathbb{Z}_6$.

Definitions

- The ***clique number*** is the size of the biggest complete subgraph in a graph.
- The ***chromatic number*** χ is the smallest amount of colors necessary to color the graph such that no adjacent vertices are the same color.

Outline

1. Definitions
   - Commutative Rings
   - Graph Theory Definitions
2. Main Results
   - Connectivity and Diameter
   - Clique and Chromatic Number
3. The Counterexample

Connectivity and Diameter

Theorem

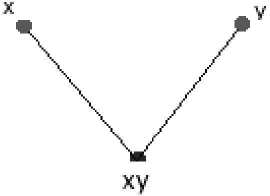
$\Gamma(R)$ is connected with a diameter less than or equal to three.

Case 1: Let $xy = 0$.

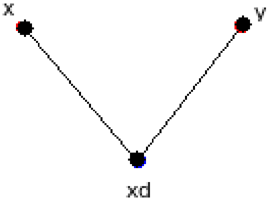
Connectivity and Diameter cont.

Case 2: Let $xy \neq 0$.

Subcase A: $x^2 = 0, y^2 = 0$



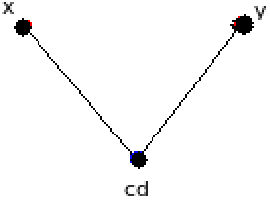
Subcase B: $x^2 = 0, y^2 \neq 0, yd = 0$



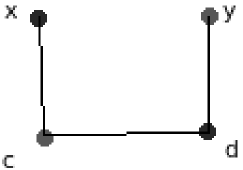
Connectivity and Diameter cont.

Case 2 cont.: Let $xy \neq 0$.

Subcase C: $x^2 \neq 0, y^2 \neq 0, xc = 0, yd = 0, cd \neq 0$



Subcase D: $x^2 \neq 0, y^2 \neq 0, xc = 0, yd = 0, cd = 0$



Outline

1 Definitions
- Commutative Rings
- Graph Theory Definitions

2 Main Results
- Connectivity and Diameter
- Clique and Chromatic Number

3 The Counterexample

The Goal

Theorem

Let $n = {p_1}^{2n_1}{p_2}^{2n_2}...{p_k}^{2n_k}{q_1}^{2m_1+1}{q_2}^{2m_2+1}...{q_r}^{2m_r+1}$ be the prime factorization of n.

$cl(\Gamma(\mathbb{Z}_n)) = \chi(\Gamma(\mathbb{Z}_n)) = {p_1}^{n_1}{p_2}^{n_2}...{p_k}^{n_k}{q_1}^{m_1}{q_2}^{m_2}...{q_r}^{m_r} + r - 1$

First Step: $\mathbb{Z}_{p^{2n}}$

Let $S = \{ap^i | n \leq i < 2n, (p, a) = 1, a \neq 0\}$.

Lemma

S forms the biggest complete subgraph of $\Gamma(\mathbb{Z}_{p^{2n}})$.

Proof.

1. Suppose ap^i with $i < n$ can be added to S.
   - Thus, $ap^i bp^n = abp^{i+n}$.
   - However, $i + n < 2n$.

   $\rightarrow\leftarrow$
2. Suppose ap^i and bp^j with $i, j < n$ are part of a different complete subgraph that is bigger than S.
   - Then $ap^i bp^j = abp^{i+j}$.
   - But $i + j < 2n$.

   $\rightarrow\leftarrow$

Hence, $|S|$ is the clique number of $\Gamma(\mathbb{Z}_{p^{2n}})$. □

First Step cont.

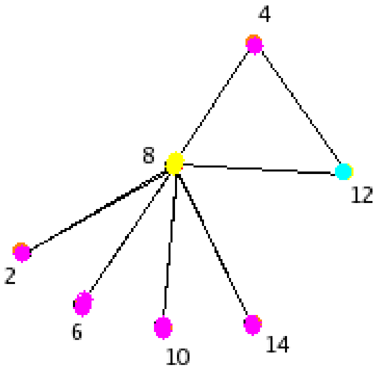
Lemma

$\mathrm{cl}(\Gamma(\mathbb{Z}_{p^{2n}})) = \chi(\Gamma(\mathbb{Z}_{p^{2n}}))$

Proof.

χ must be at least the clique number. None of the remaining vertices are adjacent to the vertex $v = p^n$, so they are all the same color as v. Thus, $\mathrm{cl}(\Gamma(\mathbb{Z}_{p^{2n}})) = \chi(\Gamma(\mathbb{Z}_{p^{2n}}))$. □

Example Graph



Zero divisor graph of $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$

First Step cont.

Theorem

$\mathrm{cl}(\Gamma(Z_{p^{2n}})) = \chi(\Gamma(Z_{p^{2n}})) = p^n - 1$

Proof.

Each element of S is a multiple of p^n, so we can count these elements by their coefficients: $1p^n$, $2p^n$,...,$p^n p^n$. This gives us p^n elements of S until we subtract 1 for $p^n p^n = 0$, which is not in the zero divisor graph. □

The Odd Case: $\mathbb{Z}_{q^{2m+1}}$

- We used the same method for $\mathbb{Z}_{q^{2m+1}}$, but had to make a few adjustments.
- For $\mathbb{Z}_{q^{2m+1}}$, $S = \{aq^i | m+1 \leq i < 2m+1, (q,a) = 1, a \neq 0\} \bigcup \{q^m\}$.
- The only difference in the two proofs comes about because of the $\{q^m\}$. This causes the counting number to be q^m.
- We used the same method for $\mathbb{Z}_{p^{2n}} \times \mathbb{Z}_{q^{2m}}$, $\mathbb{Z}_{p^{2n}} \times \mathbb{Z}_{q^{2m+1}}$, and $\mathbb{Z}_{p^{2n+1}} \times \mathbb{Z}_{q^{2m+1}}$ with the necessary adjustments.

General Case

$$\begin{aligned}\mathbb{Z}_n &= \mathbb{Z}_{p_1^{2n_1} p_2^{2n_2} \dots p_k^{2n_k} q_1^{2m_1+1} q_2^{2m_2+1} \dots q_r^{2m_r+1}} \\ &= \mathbb{Z}_{p_1^{2n_1}} \times \mathbb{Z}_{p_2^{2n_2}} \times \dots \times \mathbb{Z}_{p_k^{2n_k}} \times \mathbb{Z}_{q_1^{2m_1+1}} \times \mathbb{Z}_{q_2^{2m_2+1}} \times \dots \times \mathbb{Z}_{q_r^{2m_r+1}}\end{aligned}$$

$$\begin{aligned}S = \ & \{(a_1 {p_1}^{i_1}, a_2 {p_2}^{i_2}, \dots, a_k {p_k}^{i_k}, b_1 {q_1}^{j_1}, b_2 {q_2}^{j_2}, \dots, b_r {q_r}^{j_r})| \\ & n_l \leq i_l \leq 2n_l, a < p, a \neq 0, m_l + 1 \leq j_l \leq 2m_l + 1, b < q, b \neq 0\} \\ & \bigcup \{(0, 0, \dots, q_1^{m_1}, 0, \dots, 0)\} \bigcup \dots \bigcup \{(0, 0, \dots, 0, q_r^{m_r})\} \\ & - \{(0, 0, \dots, 0)\}\end{aligned}$$

We solve the general case by combining the methods we used to solve the previous cases.

General Proof

Theorem

Let $n = {p_1}^{2n_1}{p_2}^{2n_2}...{p_k}^{2n_k}{q_1}^{2m_1+1}{q_2}^{2m_2+1}...{q_r}^{2m_r+1}$ be the prime factorization of n.

$cl(\Gamma(\mathbb{Z}_n)) = \chi(\Gamma(\mathbb{Z}_n)) = p_1^{n_1}p_2^{n_2}...p_k^{n_k}q_1^{m_1}q_2^{m_2}...q_r^{m_r} + r - 1$

Proof

Since $n = {p_1}^{2n_1}{p_2}^{2n_2}...{p_k}^{2n_k}{q_1}^{2m_1+1}{q_2}^{2m_2+1}...{q_r}^{2m_r+1}$, we have $\mathbb{Z}_n = \mathbb{Z}_{p_1^{2n_1}} \times \mathbb{Z}_{p_2^{2n_2}} \times ... \times \mathbb{Z}_{p_k^{2n_k}} \times \mathbb{Z}_{q_1^{2m_1+1}} \times \mathbb{Z}_{q_2^{2m_2+1}} \times ... \times \mathbb{Z}_{q_r^{2m_r+1}}$ as a result of the Chinese Remainder Theorem. We use this as well as our 2 base cases for the proof.

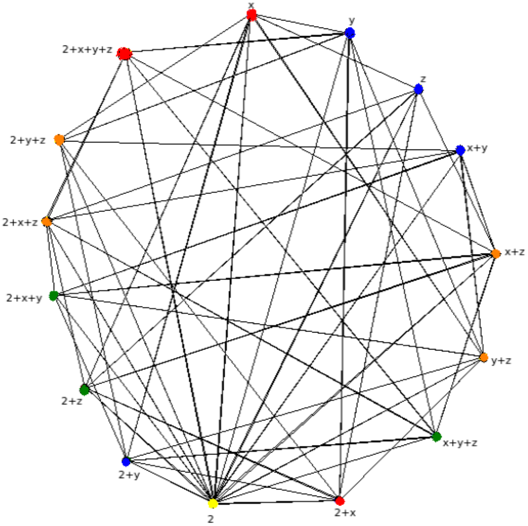
The Counterexample

- While $cl(Z_n) = \chi(Z_n)$, this same equation does not hold for all commutative rings.
- The ring $R = \mathbb{Z}_4[X, Y, Z]$ is the polynomials over $\mathbb{Z}_4$ (an infinite ring).
- We mod by a particular (messy!) ideal I to form the ring $R/I = T$. (This incorporates lots of extra relations to T.)
- After all the relations, $|R/I = T| = 32$.
- It turns out that $cl(\Gamma(T)) = 4$ and $\chi(\Gamma(T)) = 5$.

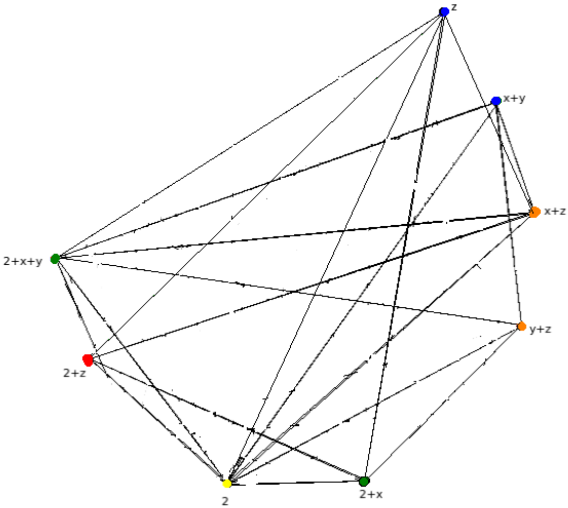
The Multiplication

- The elements of T are of the form $a + bx + cy + dz$ such that $a, b, c, d \in \mathbb{Z}_4$.
- For example, two elements of T are $x + z$ and $2 + y$.
- When multiplied together, $(x + z) \times (2 + y) = 2x + 2z + xy + yz$. With our given relations, this is equal to 2.

The Zero Divisor Graph



The Zero Divisor Graph: A Closer Look



Conclusion

There are at least three other papers written on this subject. The connected/diameter proof is in David Anderson's and Livingston's work. Beck conjectured that $cl(\mathbb{Z}_n) = \chi(\mathbb{Z}_n)$ for all rings, and Dan Anderson and Naseer's work proves Beck wrong.

THANK YOU!!!